

On weight choosability and additive choosability numbers of graphs

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Abstract

In 2004, Karoński, Łuczak, and Thomason conjectured that the edges of any connected graph on at least 3 vertices may be weighted from the set $\{1, 2, 3\}$ so that the vertices are properly coloured by the sums of their incident edge weights. A subsequent conjecture by Przybyło and Woźniak (2010) states that weights from $\{1, 2\}$ suffice if one also weights the vertices of the graph. Bartnicki, Grytczuk and Niwcyk (2009), Przybyło and Woźniak (2011), and Wong, Yang and Zhu (2010) introduced list variations of these weightings. In this paper, Alon's Combinatorial Nullstellensatz is used to prove the first known bounds on the list sizes required to ensure a colouring by sums exists for edge weightings and total weightings. Some constructive results on list variation of additive colourings, a concept introduced by Czerwiński, Grytczuk, and Żelazny (2009), are also presented.

1 Introduction

A graph $G = (V, E)$ will be simple and loopless unless otherwise stated. An **edge k -weighting**, w , of G is an assignment of a number from $[k] := \{1, 2, \dots, k\}$ to each $e \in E(G)$, that is $w : E(G) \rightarrow [k]$. Karoński, Łuczak, and Thomason [8] conjecture that, for every graph without a component isomorphic to K_2 , there is an edge 3-weighting such the function $S : V(G) \rightarrow \mathbb{Z}$ given by $S(v) = \sum_{e \ni v} w(e)$ is a proper colouring of $V(G)$ (in other words, any two adjacent vertices have different sums of incident edge weights). If such a proper colouring S exists, then w is a **vertex colouring by sums**. Let $\chi_{\Sigma}^e(G)$ the smallest value of k such that a graph

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G has an edge k -weighting which is a vertex colouring by sums. A graph G is **nice** if no component is isomorphic to K_2 . Karoński, Łuczak, and Thomason's conjecture (frequently called the "1-2-3 Conjecture") may be expressed as follows:

1-2-3 Conjecture. *If G is a nice graph, then $\chi_\Sigma^e(G) \leq 3$.*

The best known upper bound on $\chi_\Sigma^e(G)$ is due to Kalkowski, Karonski and Pfender [7], who show that $\chi_\Sigma^e(G) \leq 5$ if G is nice.

In [3], Bartnicki, Grytczuk and Niwczyk consider a "list variation" of the 1-2-3 Conjecture. Assign to each edge $e \in E(G)$ a list of k real numbers, say L_e , and choose a weight $w(e) \in L_e$ for each $e \in E(G)$. The resulting function $w : E(G) \rightarrow \cup_{e \in E(G)} L_e$ is called an **edge k -list-weighting**. Given a graph G , the smallest k such that any assignment of lists of size k to $E(G)$ permits an edge k -list-weighting which is a vertex colouring by sums is denoted $\text{ch}_\Sigma^e(G)$ and called the **edge weight choosability number** of G . The following, stronger, conjecture is proposed in [3]:

List 1-2-3 Conjecture. *If G is a nice graph, then $\text{ch}_\Sigma^e(G) \leq 3$.*

There is no known constant K such that $\text{ch}_\Sigma^e(G) \leq K$ for any nice graph G . However, the analogous problem for digraphs is solved in [3], where a constructive method is used to show that $\text{ch}_\Sigma^e(D) \leq 2$ for any digraph D . An alternate proof which uses Alon's Combinatorial Nullstellensatz [2] may be found in [9].

One may also consider edge list-weightings which colour vertices by products rather than sums. For a graph G , the smallest k such that any assignment of lists of size k to $E(G)$ permits an edge k -list-weighting which is a vertex colouring by products is denoted $\text{ch}_\Pi^e(G)$. Note that we exclude 0 from being included in any list, since it is unreasonable to use 0 to weight an edge (each end will receive colour 0 and hence the colouring will not be proper). If one wished to obtain a colouring by products, one could take a logarithm of the absolute value of every value in every list and look for a colouring by sums instead. This observation leads us to the following relationship between $\text{ch}_\Sigma^e(G)$ and $\text{ch}_\Pi^e(G)$:

Proposition 1.1. *If G is a nice graph, then $\text{ch}_\Pi^e(G) \leq 2\text{ch}_\Sigma^e(G)$.*

Another generalization of the 1-2-3 Conjecture allows each vertex $v \in V(G)$ to receive a weight $w(v)$; the colour of v is then $w(v) + \sum_{e \ni v} w(e)$ rather than simply $\sum_{e \ni v} w(e)$. Such a function $w : V \cup E \rightarrow [k]$ is called a **total k -weighting**. The smallest k for which G has a total k -weighting that is a proper colouring by sums is denoted $\chi_\Sigma^t(G)$. A similar list generalization as above may be considered; the smallest k such that the list version holds is denoted $\text{ch}_\Sigma^t(G)$. The following two conjectures are made in [12] and [13] & [14] respectively:

1-2 Conjecture. *If G is any graph, then $\chi_{\Sigma}^t(G) \leq 2$.*

List 1-2 Conjecture. *If G is any graph, then $\text{ch}_{\Sigma}^t(G) \leq 2$.*

Though the 1-2 Conjecture remains open, Przybyło [11] has shown that a total weighting w of G which properly colours $V(G)$ by sums always exists with $w(v) \in \{1, 2\}$ and $w(e) \in \{1, 2, 3\}$ for all $v \in V(G), e \in E(G)$.

In [14], Wong and Zhu study (k, l) -**total list-assignments**, which are assignments of lists of size k to the vertices of a graph and lists of size l to the edges. If any (k, l) -total list-assignment of G permits a total weighting which is a vertex colouring by sums, then G is (k, l) -**weight choosable**. Obviously, if a graph G is (k, l) -weight choosable, then $\text{ch}_{\Sigma}^t(G) \leq \max\{k, l\}$. The conjectures made in [14] are strengthened versions of the List 1-2-3 Conjecture and List 1-2 Conjectures – namely that every graph is $(2, 2)$ -weight choosable and every graph with no edge component is $(1, 3)$ -weight choosable. The most current work on (k, l) -weight choosability focuses on graph classes which are (k, l) -weight choosable for small values of k and l (see [3], [13], [15], [14]). Table 1 summarizes these results, most of which are proven using a Combinatorial Nullstellensatz approach introduced by Bartnicki et al. [3].

| Type of graph | (k, l) -weight choosability |
|--------------------------|-------------------------------|
| K_2 | $(2, 1)$ |
| $K_n, n \geq 3$ | $(1, 3), (2, 2)$ |
| $K_{n,m}, n \geq 2$ | $(1, 2)$ |
| $K_{m,n,1,\dots,1}$ | $(2, 2)$ |
| trees | $(1, 3), (2, 2)$ |
| unicyclic graphs | $(2, 2)$ |
| generalized theta graphs | $(1, 3), (2, 2)$ |
| wheels | $(2, 2)$ |

Table 1: (k, l) -weight choosability of classes of graphs

One can also trivially see that every graph G is $(\text{ch}(G), 1)$ -weight choosable, and hence $\text{ch}_{\Sigma}^t(G) \leq \text{ch}(G)$ (so, for example, $\text{ch}_{\Sigma}^t(G) \leq 5$ for any planar graph G).

In this paper, the methods used in [3] are extended to establish the first nontrivial general bounds on $\text{ch}_{\Sigma}^e(G)$ and $\text{ch}_{\Sigma}^t(G)$ which hold for every admissible graph G .

2 The permanent method and Alon's Nullstellensatz

Let $G = (V, E)$ be a graph, with $E(G) = \{e_1, \dots, e_m\}$ and $V(G) = \{v_1, \dots, v_n\}$. Associate with each e_i the variable x_i and with each v_j the variable x_{m+j} . Define two more variables for each $v_j \in V(G)$: $X_{v_j} = \sum_{e_i \ni v_j} x_i$ and $Y_{v_j} = x_{m+j} + X_{v_j}$. For an orientation D of G , define the following two polynomials, where $l = m + n$:

$$P_D(x_1, \dots, x_m) = \prod_{(u,v) \in E(D)} (X_v - X_u)$$

$$T_D(x_1, \dots, x_l) = \prod_{(u,v) \in E(D)} (Y_v - Y_u).$$

Let w be an edge weighting of G . By letting $x_i = w(e_i)$ for $1 \leq i \leq m$, w is a proper vertex colouring by sums if and only if $P_D(w(e_1), \dots, w(e_m)) \neq 0$. A similar conclusion can be made about T_D if w is a total weighting of G .

This leads us to the problem of determining when the polynomials P_D and T_D do not vanish everywhere, i.e., when there exist values of the variables for which the polynomial is non-zero. Alon's famed Combinatorial Nullstellensatz gives sufficient conditions to guarantee that a polynomial does not vanish everywhere.

Combinatorial Nullstellensatz (Alon [2]). *Let \mathbb{F} be an arbitrary field, and let $f = f(x_1, \dots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$. Suppose the total degree of f is $\sum_{i=1}^n t_i$, where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^n x_i^{t_i}$ in f is nonzero. If S_1, \dots, S_n are subsets of \mathbb{F} with $|S_i| > t_i$, then there are $s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n$ so that*

$$f(s_1, \dots, s_n) \neq 0.$$

For a polynomial $P \in \mathbb{F}[x_1, \dots, x_l]$ and a monomial term M of P , let $h(M)$ be the largest exponent of any variable in M . The **monomial index** of P , denoted $\text{mind}(P)$, is the minimum $h(M)$ taken over all monomials of P . Define the graph parameters $\text{mind}(G) := \text{mind}(P_D)$ and $\text{tmind}(G) := \text{mind}(T_D)$, where D is an orientation of G . Note that, given a graph G and two orientations D and D' , $P_D(x_1, \dots, x_l) = \pm P_{D'}(x_1, \dots, x_l)$; a similar argument holds for T_D . The parameters $\text{mind}(G)$ and $\text{tmind}(G)$ are hence well-defined. Note that, for any graph G , $\text{tmind}(G) \leq \text{mind}(G)$.

The following lemma is obtained by applying the Combinatorial Nullstellensatz to P_D and T_D :

Lemma 2.1. *Let G be a graph and k a positive integer.*

1. (Bartnicki, Grytczuk, Niwczyk [3]) *If G is nice and $\text{mind}(G) \leq k$, then $\text{ch}_\Sigma^e(G) \leq k + 1$.*
2. (Przybyło, Woźniak [13]) *If $\text{tmind}(G) \leq k$, then $\text{ch}_\Sigma^t(G) \leq k + 1$.*

The following proposition will also be useful.

Proposition 2.2. *If G is a graph with connected components G_1, G_2, \dots, G_k , then $\text{mind}(G) = \max\{\text{mind}(G_i) : 1 \leq i \leq k\}$.*

In [3], Bartnicki et al. show how one may study the permanent of particular $\{-1, 0, 1\}$ -matrices in order to gain insight on $\text{mind}(G)$ and $\text{tmind}(G)$. Let $\mathbb{M}(m, n)$ denote the set of all real valued matrices with m rows and n columns, and $\mathbb{M}(m)$ denote the set of square $m \times m$ matrices. The permanent of a matrix $A \in \mathbb{M}(m)$, denoted $\text{per } A$, is calculated as follows:

$$\text{per } A = \sum_{\sigma \in S_m} \prod_{i=1}^m a_{i, \sigma(i)}.$$

The permanent may also be defined for a general matrix $A \in \mathbb{M}(m, n)$ if $n \geq m$. Let $Q_{m,n}$ denote the set of sequences of length m with entries from $[n]$ which contain no repetition of elements; such sequences are also known as **m -permutations** from $[n]$. For example, $Q_{2,3} = \{(1, 2), (1, 3), (2, 1), (2, 3), (3, 1), (3, 2)\}$. The permanent of A is defined as follows:

$$\text{per } A = \sum_{\alpha \in Q_{m,n}} \prod_{i=1}^m a_{i, \alpha(i)} = \sum_{i=1}^{\binom{n}{m}} \text{per } B_i,$$

where $\{B_i \mid 1 \leq i \leq \binom{n}{m}\}$ is the set of all $m \times m$ submatrices of A .

The **permanent rank** of a matrix A (not necessarily square) is the size of the largest square submatrix of A having nonzero permanent. Let $A^{(k)} = [AA \cdots A]$ denote the matrix formed of k consecutive copies of A . If A has size $m \times l$, then the **permanent index** of A is the smallest k , if it exists, such that $A^{(k)}$ has permanent rank m . This parameter is denoted $\text{pind}(A)$. If such a k does not exist, then $\text{pind}(A) := \infty$. Alternately, $\text{pind}(A)$ is the smallest k such that a square matrix of size m having nonzero permanent can be constructed by taking columns from A , each column taken no more than k times.

There are three matrices related to directed graphs which will be of interest:

Definition 2.3. Let $G = (V, E)$ be a graph, $V(G) = \{v_1, \dots, v_n\}$, $E(G) = \{e_1, \dots, e_m\}$. For an orientation D of G , define the matrices $A_D \in \mathbb{M}(m)$, $B_D \in \mathbb{M}(m, n)$, and $M_D \in \mathbb{M}(m, m+n)$ as follows:

- $A_D = (a_{i,j})$ where $a_{i,j} = \begin{cases} 1 & \text{if } e_j \text{ is incident with the head of } e_i \\ -1 & \text{if } e_j \text{ is incident with the tail of } e_i \\ 0 & \text{otherwise} \end{cases}$
- $B_D = (b_{i,j})$ where $b_{i,j} = \begin{cases} 1 & \text{if } v_j \text{ is the head of } e_i \\ -1 & \text{if } v_j \text{ is the tail of } e_i \\ 0 & \text{otherwise} \end{cases}$
- $M_D = (A_D \mid B_D)$.

The following lemmas, which relate the matrices A_D , B_D , and M_D to the polynomials P_D and T_D , provide the fundamental link between the graphic polynomials of interest and matrix permanents:

Lemma 2.4 (Bartnicki, Grytczuk, Niwczyk [3]). Let $A = (a_{ij}) \in \mathbb{M}(m)$ have finite permanent index. If $P(x_1, \dots, x_m) = \prod_{i=1}^m (a_{i1}x_1 + \dots + a_{im}x_m)$, then $\text{mind}(P) = \text{pind}(A_D)$.

The proof is omitted, but the result follows from the fact that the coefficient of $x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}$ in the expansion of P is equal to $\frac{\text{per}(M)}{k_1! \dots k_m!}$ where M is the $m \times m$ matrix where column a_j from A appears k_j times. Lemma 2.4 immediately implies the following vital link between the (total) monomial index of a graph G and the permanent index of A_D (respectively, T_D) for any orientation D of G :

Lemma 2.5. Let D be an orientation of a graph G .

1. (Bartnicki, Grytczuk, Niwczyk [3]) If G is nice, then $\text{mind}(G) = \text{pind}(A_D)$.
2. (Przybyło, Woźniak [13]) For any graph G , $\text{tmind}(G) = \text{pind}(M_D)$.

Lemmas 2.1 and 2.5 imply the following relationship between $\text{ch}_\Sigma^e(G)$ and A_D , and $\text{ch}_\Sigma^t(G)$ and T_D :

Corollary 2.6. Let G be a graph, D an orientation of G , and k a positive integer.

1. If G is nice and $\text{pind}(A_D) \leq k$, then $\text{ch}_\Sigma^e(G) \leq k + 1$.
2. If $\text{pind}(M_D) \leq k$, then $\text{ch}_\Sigma^t(G) \leq k + 1$.

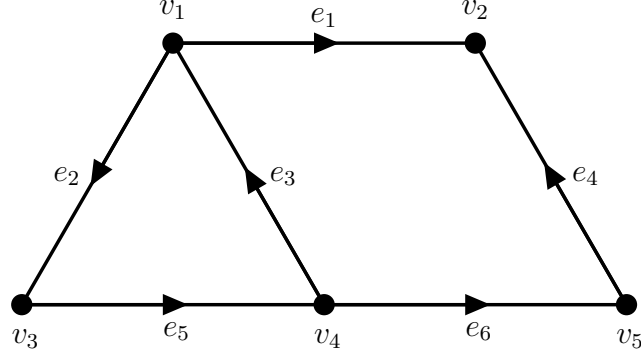


Figure 1: A digraph used to illustrate A_D , B_D , and M_D

Consider the following illustrative example. Let D be the digraph in Figure 1 and let G be its underlying simple graph. The associated polynomial, P_D , is

$$\begin{aligned}
P_D(x_1, \dots, x_6) &= (x_1 + x_4 - x_1 - x_2 - x_3) \times (x_2 + x_5 - x_1 - x_2 - x_3) \\
&\quad \times (x_1 + x_2 + x_3 - x_3 - x_5 - x_6) \times (x_1 + x_4 - x_4 - x_6) \\
&\quad \times (x_3 + x_5 + x_6 - x_2 - x_5) \times (x_4 + x_6 - x_3 - x_5 - x_6) \\
&= (x_4 - x_2 - x_3) \times (x_5 - x_1 - x_3) \times (x_1 + x_2 - x_5 - x_6) \\
&\quad \times (x_1 - x_6) \times (x_3 + x_6 - x_2) \times (x_4 - x_3 - x_5).
\end{aligned}$$

Recalling the definition of M_D , note that the coefficients of each factor in P_D correspond to the entries in each row of A_D :

$$M_D = [A_D \mid B_D] = \left(\begin{array}{cccccc|ccccc} 0 & -1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \end{array} \right).$$

Since $\det A_D = -4 \neq 0$, we have $\text{pind}(A_D) = 1$ (each column from A_D is chosen once). This implies, by Lemma 2.4, that $\text{mind}(G) = 1$ (i.e. the term $x_1x_2x_3x_4x_5x_6$ has nonzero coefficient). Hence, $\text{ch}_\Sigma^e(G) \leq 2$ by the Combinatorial Nullstellensatz. Since there are adjacent vertices of equal degree in G , we have $\chi_\Sigma^e(G) \neq 1$, implying $\text{ch}_\Sigma^e(G) \neq 1$ and so $\text{ch}_\Sigma^e(G) = 2$.

3 Bounds for edge list-weightings and total list-weightings

Armed with the Combinatorial Nullstellensatz and the permanent method, we may now proceed with the major results of this chapter. The following theorem summarizes two major results which will be proven on vertex colouring list-weightings:

Theorem 3.1. *If G is a nice graph with maximum degree $\Delta(G)$, then $\text{ch}_\Sigma^e(G) \leq 2\Delta(G) + 1$. If G is any graph, then $\text{ch}_\Sigma^t(G) \leq \lceil \frac{2}{3}\Delta(G) \rceil + 1$.*

The first result gives the first known upper bound on $\text{ch}_\Sigma^e(G)$. Since $\text{ch}_\Sigma^t(G)$ is bounded above by $\text{ch}(G) \leq \Delta(G) + 1$, the second result gives an improvement on best available bound on $\text{ch}_\Sigma^t(G)$ in terms of $\Delta(G)$.

Theorem 3.1 is proven by establishing bounds on $\text{mind}(G)$ and $\text{tmind}(G)$ using the permanent method. One important tool is the following lemma, a generalization of a similar result in [3]:

Lemma 3.2 (Przybyło, Woźniak [13]). *Let A be an $m \times l$ matrix, and let L be an $m \times m$ matrix where each column of L is a linear combination of columns of A . Let n_j denote the number of columns of L in which the j^{th} column of A appears with nonzero coefficient. If $\text{per } L \neq 0$, then $\text{pind}(A) \leq \max \{n_j \mid j = 1, \dots, l\}$.*

We will also find the following theorem useful, which gives a method for constructing graphs in a way that preserves the property of having low monomial index:

Theorem 3.3 (Bartnicki, Grytczuk, Niwczyk [3]). *Let G be a simple graph with $\text{mind}(G) \leq 2$. Let U be a nonempty subset of $V(G)$. If F is a graph obtained by adding two new vertices u, v to $V(G)$ and joining them to each vertex of U , and H is a graph obtained from F by joining u and v , then $\text{mind}(F), \text{mind}(H) \leq 2$.*

As a consequence, the following graph classes have low monomial index and hence small values of $\text{ch}_\Sigma^e(G)$:

Corollary 3.4 (Bartnicki, Grytczuk, Niwczyk [3]). *If G is a complete graph, a complete bipartite graph, or tree, then $\text{mind}(G) \leq 2$ and hence $\text{ch}_\Sigma^e(G) \leq 3$.*

It can also be easily shown that the same bound holds for cycles.

Proposition 3.5. *If $G = C_n$, then $\text{mind}(G) \leq 2$ and hence $\text{ch}_\Sigma^e(G) \leq 3$.*

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$. Let D be the orientation of G with $A(D) = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$. Consider the colouring polynomial

$$P_D = (x_2 - x_n)(x_3 - x_1)(x_4 - x_2) \cdots (x_n - x_{n-2})(x_1 - x_{n-1}).$$

Since each variable appears in exactly two factors of P_D , no exponent in the expansion of P_D exceeds 2, and hence $\text{mind}(G) \leq 2$. \square

In order to prove Theorem 3.1, the following generalization of Theorem 3.3 is required:

Lemma 3.6. *Let G be a graph with finite monomial index $\text{mind}(G) \geq 1$. Let U be a nonempty subset of $V(G)$. If F is a graph obtained by adding two new vertices u, v to $V(G)$ and joining them to each vertex of U , and H is a graph obtained from F by joining u and v , then $\text{mind}(F), \text{mind}(H) \leq \max\{2, \text{mind}(G)\}$.*

The proof which follows is an adaptation of the proof of Theorem 3.3 in [3]. Given a matrix A with columns a_1, a_2, \dots, a_n and a sequence of (not necessarily distinct) column indices $K = (i_1, i_2, \dots, i_k)$, $A(K)$ is defined to be the matrix $A(K) = (a_{i_1} \ a_{i_2} \ \cdots \ a_{i_k})$.

Proof. Let $U = \{u_1, \dots, u_k\}$ be the subset of $V(G)$ stated in the theorem. Let $E_u = \{e_1, e_3, \dots, e_{2k-1}\}$ and $E_v = \{e_2, e_4, \dots, e_{2k}\}$ be the sets of edges incident to the vertices u and v , respectively. Assume that these edges are oriented toward U , and that for each $i = 1, 2, \dots, k$ the edges e_{2i-1} and e_{2i} have the same head.

Let D be an orientation of F , D' the induced orientation of G , and consider the matrices A_D and $A_{D'}$. Let A_1, \dots, A_{2k} be the first $2k$ columns of A_D , corresponding to $\{e_1, e_2, \dots, e_{2k}\}$. If we write $A = (A_1 \ \cdots \ A_{2k})$, then $A_F = (A \ B)$ where $B = \begin{pmatrix} X \\ A_{D'} \end{pmatrix}$.

Let Y be the $(2k) \times (2k)$ matrix and Z the $(|E(F)| - 2k) \times (2k)$ matrix such that $A = \begin{bmatrix} Y \\ Z \end{bmatrix}$. Since the edges e_{2i-1} and e_{2i} have the same head for each $i = 1, 2, \dots, k$, A_{2i-1} and A_{2i} agree on Z . Furthermore, Y may be written as a block matrix, where $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ occupies the diagonals and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is everywhere else, as seen in Figure 2.

There exists a matrix of columns from $A_{D'}$, with no column used more than $\text{mind}(G)$ times, with nonzero permanent. Let K denote the sequence of edges of G which index this matrix. Consider a new matrix

$$M = (M_1 \ M_1 \ M_2 \ M_2 \ \cdots \ M_k \ M_k \ B(K)),$$

where $M_j = A_{2j-1} - A_{2j}$ for $j = 1, 2, \dots, k$.

$$Y = \begin{pmatrix} 0 & 1 & -1 & 0 & \cdots & -1 & 0 \\ 1 & 0 & 0 & -1 & \cdots & 0 & -1 \\ -1 & 0 & 0 & 1 & & -1 & 0 \\ 0 & -1 & 1 & 0 & & 0 & -1 \\ & \vdots & & & \ddots & & \\ -1 & 0 & -1 & 0 & & 0 & 1 \\ 0 & -1 & 0 & -1 & & 1 & 0 \end{pmatrix}.$$

Figure 2: The block matrix Y

The properties of the columns of A outlined above imply that the matrix M can be written as follows: $M = \begin{pmatrix} R & X(K) \\ 0 & A_{D'}(K) \end{pmatrix}$, where R has all constant rows:

$$R = \begin{pmatrix} -1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & -1 & & -1 \\ & \vdots & & \ddots & \\ 1 & 1 & 1 & & 1 \end{pmatrix}.$$

Since $\text{per } M = \text{per } R \times \text{per } A_{D'}(K)$, each of $\text{per } R$ and $\text{per } A_G(K)$ are nonzero, and any column of M appears in the linear combination of at most 2 columns of R , Lemma 3.2 implies that $\text{mind}(F) \leq \max\{2, \text{mind}(G)\}$.

Now consider the matrix A_H . Let $e_0 = uv$, and orient it from v to u . The matrix A_H is precisely A_F with a row and column added for e_0 (say, as the first row and column). It can be depicted in block form $A_H = \begin{pmatrix} Y' & X' \\ Z' & A_G \end{pmatrix}$, where Y' and Z' are the matrices depicted in Figure 3 on page 10.

$$Y' = \begin{pmatrix} 0 & 1 & -1 & \cdots & 1 & -1 \\ -1 & & & & & \\ -1 & & & & & \\ \vdots & & & Y & & \\ -1 & & & & & \\ -1 & & & & & \end{pmatrix}, \quad Z' = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad Z$$

Figure 3: The matrices Y' and Z'

Let A_0, A_1, \dots, A_{2k} denote the first $2k + 1$ columns of A_H , corresponding to the

edges e_0, e_1, \dots, e_{2k} . Form a new matrix

$$N = (N_0 \ N_0 \ N_1 \ N_2 \ N_2 \ \cdots \ N_k \ N_k \ B(K)),$$

so that $N_0 = A_0$ and $N_j = A_{2j-1} - A_{2j}$ for $j = 1, 2, \dots, k$. Arguing as before, $N = \begin{pmatrix} R' & X'(K) \\ 0 & A_G(K) \end{pmatrix}$, where R' is the following square matrix:

$$R' = \begin{pmatrix} 0 & 0 & 2 & 2 & & 2 \\ -1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & -1 & 1 & 1 & & 1 \\ & \vdots & \vdots & & \ddots & \\ -1 & -1 & 1 & 1 & & 1 \end{pmatrix}.$$

It is shown in [3] that $\text{per } R' \neq 0$. Hence $\text{per } N = \text{per } R' \times \text{per } A_G(K) \neq 0$, and since any column of N appears in the linear combination of at most 2 columns of R' , Lemma 3.2 implies that $\text{mind}(H) \leq \max\{2, \text{mind}(G)\}$. \square

We may now proceed with the proof of Theorem 3.1. By carefully orienting the edges of a graph, applying the lemmas above will give us a matrix with non-zero permanent; applying the permanent method gives the result.

Theorem 3.7. *If G is a nice graph on at least 3 vertices, then $\text{mind}(G) \leq 2\Delta(G)$.*

Proof. If G is a tree, cycle, or complete graph, then $\text{mind}(G) \leq 2$ by Corollary 3.4 and Proposition 3.5, and hence the theorem holds for the following graphs: P_3 , K_3 , P_4 , $K_{1,3}$, C_4 , and K_4 . If G is isomorphic to K_3 with a leaf or C_4 with a chord, then one may check that the theorem holds for G by straightforward computation of the associated colouring polynomial P_D for any orientation D . Hence, the theorem holds for any connected graph on 3 or 4 vertices.

We proceed now by induction on $|V(G)|$. We may assume that G is connected, since Proposition 2.2 states that $\text{mind}(G)$ is at most the largest monomial index of its components. Let G be a connected graph on at least 5 vertices, and for any graph H with $|V(H)| < |V(G)|$, assume that $\text{mind}(H) \leq 2\Delta(H)$.

If G is a complete graph, then the theorem holds by Corollary 3.4. Assume that G is not complete. There exist $u, v, w \in V(G)$ such that the induced subgraph $G[\{u, v, w\}]$ is a path of length 2 (or, uvw is an induced 2-path). Choose this 2-path such that $d(u) + d(w)$ is minimum. The ultimate goal will be to apply an inductive argument to $G - \{u, w\}$, however we must concern ourselves with whether or not this

subgraph of G is nice. To this end, we define the following sets of edges:

$$\begin{aligned}\mathcal{F} &= \text{the edges of those components in } G - \{u, w\} \text{ isomorphic to } K_2 \\ E_u &= \{e \in E(G) \mid e \ni u, e \neq uv\} \\ E_w &= \{e \in E(G) \mid e \ni w, e \neq vw\} \\ E_v &= \{e \in E(G) \mid e \ni v, e \neq uv, vw\} \\ E^* &= E(G) \setminus (E_u \cup E_v \cup E_w \cup \{uv, vw\})\end{aligned}$$

The path uvw and the sets of edges E_u, E_v, E_w are shown in Figure 4.

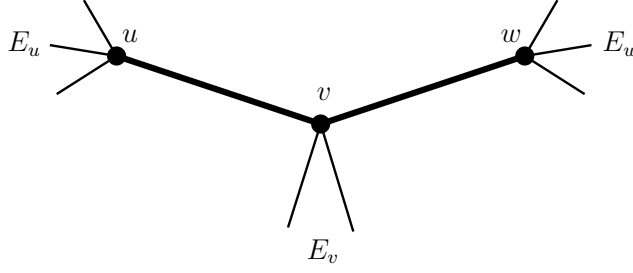


Figure 4: The induced 2-path uvw in G

Case 1: $E_v \cap \mathcal{F} \neq \emptyset$

We first consider the case that deleting u and w leaves a component isomorphic to K_2 containing the vertex v . Suppose $E_v \cap \mathcal{F} \neq \emptyset$. There can be only one edge in this intersection, otherwise the connected component containing v in $G - \{u, w\}$ would have two or more edges. This implies that, since $uv, vw \in E(G)$, we have that $N_G(v) = \{u, w, x\}$ for some vertex $x \in V(G)$. Since $\{v, x\}$ induces a graph isomorphic to K_2 in $G - \{u, w\}$, we have that $N_G(x) \subseteq \{u, v, w\}$.

If x is adjacent to both u and w , then v and x are adjacent twins. Suppose that $G \setminus \{v, x\}$ is not nice; we will show that this contradicts the choice of uvw which minimizes $d_G(u) = d_G(w)$. If G is not nice, then u, w , or both u and w are adjacent to exactly one vertex in G other than v and x ; without loss of generality, suppose that $uy \in E(G)$, $y \neq v, x$. Since $y \notin N_G(x)$, the vertices y, u, x induce a 2-path; furthermore, $d_G(y) + d_G(x) = 1 + 3 = 4$. This contradicts our choice of uvw , since $d(u) + d(w) \geq 3 + 2 = 5$. Thus, $G - \{v, x\}$ is a nice graph, and so, by Lemma 3.6, $\text{mind}(G) \leq \max\{2, \text{mind}(G - \{v, x\})\}$. By the induction hypothesis, $\text{mind}(G - \{v, x\}) \leq 2\Delta(G - \{v, x\})$, and so

$$\text{mind}(G) \leq \max\{2, \text{mind}(G - \{v, x\})\} \leq \max\{2, 2\Delta(G - \{v, x\})\} \leq 2\Delta(G).$$

We may now assume that x is not adjacent to at least one of u and w . If $w \notin N_G(x)$, then both uvw and xvw are induced 2-paths in G . By the minimality of $d(u) + d(w)$, we must have that $d(u) \leq d(v)$. If u is adjacent to x , then $d(v) = 2$ and, since u is adjacent to v as well, $d(u) = 2$ and $N_G(u) = \{v, x\}$. Otherwise, if u is not adjacent to x , then $d_G(u) = d_G(x) = 1$ and $N_G(u) = \{v\}$. In either case, u and x are twins. If $G - \{u, x\}$ is not nice, then the only edge not incident to u or x is the edge vw , contradicting our choice of G with $|V(G)| \geq 5$. Assume that $G - \{u, x\}$ is nice. By Lemma 3.6, $\text{mind}(G) \leq \max\{2, \text{mind}(G - \{u, x\})\}$, and by the induction hypothesis, $\text{mind}(G - \{u, x\}) \leq 2\Delta(G - \{u, x\})$. Thus,

$$\text{mind}(G) \leq \max\{2, \text{mind}(G - \{u, x\})\} \leq \max\{2, 2\Delta(G - \{u, x\})\} \leq 2\Delta(G).$$

If $u \notin N_G(x)$ and $w \in N_G(x)$, then the exact same argument holds as for $u \in N_G(x)$ and $w \notin N_G(x)$. Having considered all possible neighbourhoods of x , we conclude that if $E_v \cap \mathcal{F}$ is nonempty, then $\text{mind}(G) \leq 2\Delta(G)$.

Case 2: $E_v \cap \mathcal{F} = \emptyset$

Suppose that $E_v \cap \mathcal{F} = \emptyset$. The argument proceeds as follows: after choosing a “good” orientation D of G , we will construct a matrix whose columns are linear combinations of A_D with no column of A_D being used more than $2\Delta(G)$ times and with nonzero permanent. The result will then follow by Lemma 3.2.

Let D be an orientation of G where the edges of $E_u \cup \{uv\}$ and E_v are oriented toward u and v , respectively, and the edges of $E_w \cup \{vw\}$ are oriented away from w ; see Figure 5. Let c_{uv} and c_{vw} be the columns of A_D associated with the edges uv and

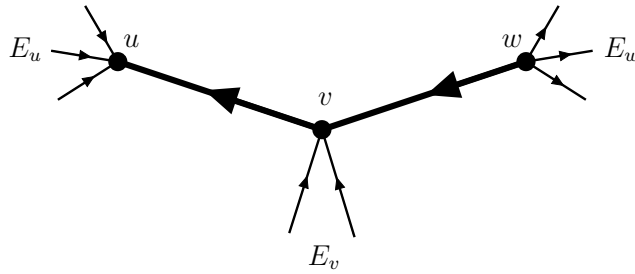


Figure 5: An orientation D of a graph G with an induced 2-path uvw

vw , respectively, and let $c = c_{uv} - c_{vw}$; see Figure 6.

We must still concern ourselves with the possibility that deleting u and w from G gives a graph which is not nice. If a component of $G - \{u, w\}$ is isomorphic to K_2 , then one vertex of this component must be adjacent to either u or w in

$$\begin{array}{c}
\begin{array}{c} uv \\ vw \\ \\ E_u \\ \\ E_w \\ \\ E_v \\ \\ E^* \end{array}
\begin{pmatrix}
c_{uv} & c_{vw} & \cdots \\
0 & -1 & \\
1 & 0 & \\
1 & 0 & \\
\vdots & \vdots & \\
1 & 0 & \\
0 & -1 & \\
\vdots & \vdots & \cdots \\
0 & -1 & \\
1 & 1 & \\
\vdots & \vdots & \\
1 & 1 & \\
0 & 0 & \\
\vdots & \vdots & \\
0 & 0 &
\end{pmatrix}
\implies c =
\begin{pmatrix}
1 \\
1 \\
1 \\
\vdots \\
1 \\
\vdots \\
1 \\
0 \\
\vdots \\
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\end{array}$$

Figure 6: An operation on two columns of A_D

G . Let $\mathcal{F} = \{f_1, \dots, f_k\}$. For each $f_i \in \mathcal{F}$, let e_i be an edge from E_u or E_w to which f_i is adjacent. Let F denote this collection of edges from $E_u \cup E_w$, and let $F_u = \{e : e \in E_u \cap F\}$ and $F_w = \{e : e \in E_w \cap F\}$.

Let $H = G - \{u, w\} - \mathcal{F}$ and $D(H)$ be the corresponding sub-digraph of D . Since we have removed all components isomorphic to K_2 , H is nice. Since H has fewer vertices than G , by the induction hypothesis, $\text{mind}(H) \leq 2\Delta(H)$. Hence, there exists a matrix L_H consisting of columns of $A_{D(H)}$, none repeated more than $2\Delta(H)$ times, with $\text{per}(L_H) \neq 0$. Let K denote the sequence of edges which indexes the columns of L_H . Recall that, for an $m \times n$ matrix A , $A^{(k)}$ is the $m \times kn$ matrix consisting of k consecutive copies of A (see page 5). Let L_G be the following block matrix:

$$L_G = \left(\begin{array}{c|c|c} c^{(d(u)+d(w))} & A_D(F) & A_D(K) \end{array} \right) = \begin{array}{c} E_u \cup E_w \cup \{uv, vw\} \\ \mathcal{F} \\ E(H) \end{array} \begin{pmatrix} J_{d(u)+d(w)} & K_1 & X_1 \\ 0 & K_2 & X_2 \\ 0 & 0 & L_H \end{pmatrix},$$

where the blocks are as follows:

- $J_{d(u)+d(w)}$ is the $(d(u) + d(w)) \times (d(u) + d(w))$ all 1's matrix.
- $K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}$ having entries in the column corresponding to $e_i \in F_u$ as follows:

- 1 in each row indexed by the other edges from E_u ,
- 1 in the row indexed by uv ,
- -1 in the row indexed by f_i , and
- 0 in all other entries.

If $e_i \in F_w$, the entries follow the same pattern with the signs swapped. Since the column associated with e_i has only one non-zero entry in the rows indexed by \mathcal{F} , K_2 is diagonal with $|F_u|$ entries being -1 and $|F_w|$ entries being 1 .

- $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, the $(|E(G)| - |E(H)|) \times |E(H)|$ submatrix of $A_D(K)$ whose rows are indexed by $E(G) \setminus E(H)$; and
- L_H , is the matrix with $\text{per}(L_H) \neq 0$ defined above.

Since $J_{d(u)+d(w)}$, K_2 , and L_H are all square matrices,

$$\begin{aligned} \text{per}(L_G) &= \text{per}(J_{d(u)+d(w)}) \cdot \text{per}(K_2) \cdot \text{per}(L_H) \\ &= (d(u) + d(w))! \cdot (-1)^{|F_u|} (1)^{|F_w|} \cdot \text{per}(L_H) \neq 0. \end{aligned}$$

Since the sets $\{uv, vw\}$, F , and $E(H)$ are pairwise disjoint, no column is used more than $\max\{d(u) + d(w), 1, \text{mind}(H)\}$ times. Lemma 3.2 states that $\text{pind}(A_D) \leq \max\{d(u) + d(w), 1, \text{mind}(H)\}$. Since $\text{pind}(A_D) = \text{mind}(G)$ (Lemma 2.5.1) and $\text{mind}(H) \leq 2\Delta(H)$ by induction,

$$\text{mind}(G) \leq \max\{d(u) + d(w), 1, \text{mind}(H)\} \leq \max\{2\Delta(G), 1, 2\Delta(H)\} \leq 2\Delta(G). \quad \square$$

Theorem 3.8. *If G is any graph, then $\text{tmind}(G) \leq \lceil \frac{2}{3}\Delta(G) \rceil$.*

Proof. The proof follows as that of Theorem 3.7, with a different matrix of interest. Assume that, for any graph H with $|V(H)| < |V(G)|$, $\text{tmind}(H) \leq \lceil \frac{2}{3}\Delta(H) \rceil$. Choose an induced 2-path uvw as before, and assume that $d(u) \leq d(w)$. Let L'_G be the following matrix consisting of linear combinations of columns from M_D for the orientation D of G :

$$L'_G = \left(c_u^{(r)} \mid c_w^{(s)} \mid c^{(t)} \mid A_D(F) \mid A_D(K) \right),$$

where

$$r = \min \left\{ d(u), \left\lceil \frac{d(u) + d(w)}{3} \right\rceil \right\}, \quad s = \left\lceil \frac{d(u) + d(w)}{3} \right\rceil, \quad t = d(u) + d(w) - r - s.$$

Let A^* be the following matrix:

$$A^* = \begin{matrix} & \begin{matrix} c_u^{(r)} & c_w^{(s)} & c^{(t)} \end{matrix} \\ \begin{matrix} uv \\ E_u \\ vw \\ E_w \end{matrix} & \begin{pmatrix} 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 \\ \vdots & \vdots & \vdots \\ 1 \cdots 1 & 0 \cdots 0 & 1 \cdots 1 \\ 0 \cdots 0 & -1 \cdots -1 & 1 \cdots 1 \\ 0 \cdots 0 & -1 \cdots -1 & 1 \cdots 1 \\ \vdots & \vdots & \vdots \\ 0 \cdots 0 & -1 \cdots -1 & 1 \cdots 1 \end{pmatrix} \end{matrix}$$

The matrix L'_G can then be rewritten as a block matrix:

$$L'_G = \begin{matrix} E_u \cup E_w \cup \{uv, vw\} \\ \mathcal{F} \\ E(H) \end{matrix} \begin{pmatrix} A^* & K_1 & X'_1 \\ 0 & K_2 & X'_2 \\ 0 & 0 & L'_H \end{pmatrix},$$

where K_1 and K_2 are defined as in the proof of Theorem 3.7 and L'_H is the matrix guaranteed by the induction hypothesis with non-zero permanent and no column repeated more than $\text{pind}(M_{D(H)}) = \text{tmind}(H) \leq \left\lceil \frac{2\Delta(H)}{3} \right\rceil$ times (recall that Lemma 2.5.2 states that $\text{pind}(M_{D(H)}) = \text{tmind}(H)$).

The permanent of A^* is $\text{per}(A^*) = r! \binom{d(u)}{r} \times s! \binom{d(w)}{s} (-1)^s \times t! \neq 0$. Since A^* , K_2 , and L'_H are all square matrices,

$$\begin{aligned} \text{per}(L'_G) &= \text{per}(A^*) \cdot \text{per}(K_2) \cdot \text{per}(L_H) \\ &= \text{per}(A^*) \cdot (-1)^{|F_u|} (1)^{|F_w|} \cdot \text{per}(L_H) \neq 0. \end{aligned}$$

We now consider the number of times the columns, c_u , c_w , and c appear in A^* . If $r = d(u)$, then $t = d(w) - \left\lceil \frac{d(u)+d(w)}{3} \right\rceil \leq d(w) - \frac{d(w)}{3} \leq \left\lceil \frac{2d(w)}{3} \right\rceil$. If $r = \left\lceil \frac{d(u)+d(w)}{3} \right\rceil$, then $t = d(u) + d(w) - 2 \left\lceil \frac{d(u)+d(w)}{3} \right\rceil \leq \frac{d(u)+d(w)}{3} \leq \left\lceil \frac{2d(w)}{3} \right\rceil$. In either case, $\max\{r, s, t\} \leq \left\lceil \frac{2d(w)}{3} \right\rceil$. Since the sets F and $E(H)$ are pairwise disjoint, no column of M_D is used more than $\max\left\{\left\lceil \frac{2d(w)}{3} \right\rceil, 1, \text{tmind}(H)\right\}$ times. Lemma 3.2

states that $\text{pind}(M_D) \leq \max \left\{ \left\lceil \frac{2d(w)}{3} \right\rceil, 1, \text{tmind}(H) \right\}$, and so

$$\begin{aligned} \text{tmind}(G) &\leq \max \left\{ \left\lceil \frac{2d(w)}{3} \right\rceil, 1, \text{tmind}(H) \right\} \leq \max \left\{ \left\lceil \frac{2\Delta(G)}{3} \right\rceil, \left\lceil \frac{2\Delta(H)}{3} \right\rceil \right\} \\ &\leq \left\lceil \frac{2\Delta(G)}{3} \right\rceil. \end{aligned} \quad \square$$

The proof of Theorem 3.1 easily follows.

Theorem 3.1. *If G is a nice graph with maximum degree $\Delta(G)$, then $\text{ch}_\Sigma^e(G) \leq 2\Delta(G) + 1$. If G is any graph, then $\text{ch}_\Sigma^t(G) \leq \lceil \frac{2}{3}\Delta(G) \rceil + 1$.*

Proof. Lemma 2.1 states that $\text{ch}_\Sigma^e(G) \leq \text{mind}(G) + 1$ and $\text{ch}_\Sigma^t(G) \leq \text{tmind}(G) + 1$; applying Lemma 2.1 to Theorems 3.7 and 3.8 gives the desired result. \square

By applying Proposition 1.1, we obtain the following bound on $\text{ch}_\Pi^e(G)$:

Theorem 3.9. *If G is a nice graph with maximum degree $\Delta(G)$, then $\text{ch}_\Pi^e(G) \leq 4\Delta(G) + 2$.*

In the proof of Theorem 3.7, no column corresponding to a vertex of the graph is chosen. As such, Theorems 3.7 and 3.8 imply the following results on (k, l) -weight choosability:

Corollary 3.10. *If G is a nice graph, then it is $(1, 2\Delta(G) + 1)$ -weight choosable.*

Corollary 3.11. *If G is any graph, then it is $(\lceil \frac{2}{3}\Delta(G) \rceil + 1, \lceil \frac{2}{3}\Delta(G) \rceil + 1)$ -weight choosable.*

By following the proof of Theorem 3.8 and letting $r = s = 1$ and $t = d(u) + d(w) - 2$, the following result is similarly obtained:

Corollary 3.12. *If G is a nice graph, then it is $(2, 2\Delta(G) - 1)$ -weight choosable.*

Recall that Przybyło's proof that $\chi_\Sigma^t(G) \leq 3$ [11] in fact shows that weights from $\{1, 2\}$ for the vertices and $\{1, 2, 3\}$ for the edges of a graph admit a weighting which is a proper vertex colouring by sums. Corollary 3.12 is thus particularly relevant, since a natural approach to the List 1-2 Conjecture is to ask for the smallest k such that every graph is $(2, k)$ -weight choosable.

4 Colouring graph products by list-weightings

Though the bounds presented in Section 3 represent progress on the List 1-2-3 and List 1-2 Conjectures, there is still room for improvement. As such, we will consider some classes of graphs where smaller upper bounds can be obtained.

The following lemma provides a decomposition approach for the classes of graphs which we will study:

Lemma 4.1. *Let G be a graph, and let H be an induced subgraph of G containing a 2-factor. Let X be a minimal edge cut separating $V(H)$ from $V(G) \setminus V(H)$. If the components of $G - H - X$ are C_1, \dots, C_k , then*

$$\begin{aligned} \text{mind}(G) &\leq \max\{|X| + \text{mind}(H), \text{mind}(G - X)\} \\ &= \max\{\text{mind}(H) + |X|, \text{mind}(C_1), \dots, \text{mind}(C_k)\}. \end{aligned}$$

Proof. Let $|V(H)| = v$ and $F = \{e_1, \dots, e_v\}$ be a 2-factor of H . Let D be an orientation of G such that the cycles of F are directed. Define the column vector $c = \sum_{i=1}^v c_i$ where c_i is the column of A_D corresponding to e_i . For each $e \in E(H) \setminus F$ there are two edges of F incident to each of the head and tail of e , and for each $e \in F$ there is one edge of F incident to each of the head and tail of e . Hence, the entries of c are nonzero in the rows indexed by the edges of X and 0 in all other entries.

There exists a matrix L_{G-X} consisting of columns of A_{G-X} with no column of A_D repeated more than $\text{mind}(G - X)$ times and $\text{per}(L_{G-X}) \neq 0$. Let K denote the sequence of edges of $G - X$ which index A_{G-X} . Consider the following matrix:

$$L = \begin{pmatrix} c^{(|X|)} & A_D(K) \end{pmatrix} = \begin{pmatrix} M & N \\ 0 & L_{G-X} \end{pmatrix},$$

where $(M \ N)$ is indexed by X , each row of M is constant, and every entry of M is nonzero. Any column indexed by $e \in E(G) \setminus F$ is used at most $\text{mind}(G - X)$ times in the construction of L , and any edge from F is used at most $|X| + \text{mind}(H)$ times. Clearly, $\text{per}(L) = \text{per}(M)\text{per}(L_{G-X}) \neq 0$, and hence $\text{mind}(G) = \max\{|X| + \text{mind}(H), \text{mind}(G - X)\}$. Since $\text{mind}(G - X) = \max\{\text{mind}(C_1), \dots, \text{mind}(C_k), \text{mind}(H)\}$ by Proposition 2.2, the result follows. \square

Recall that the Cartesian product of two graphs G and H , denoted by $G \square H$, is defined as the graph having vertex set $V(G) \times V(H)$ where two vertices (u, u') and (v, v') are adjacent if and only if either $u = v$ and u' is adjacent to v' in H or $u' = v'$ and u is adjacent to v in G . Some results on $\chi_{\Sigma}^e(G)$ for Cartesian products of graphs

are given in [10]; for instance, if G and H are regular and bipartite, then $\chi_\Sigma^e(K_n \square H)$, $\chi_\Sigma^e(C_t \square H)$, and $\chi_\Sigma^e(G \square H)$ are at most 2 for $n \geq 4$, and $t \geq 4, t \neq 5$. Lemma 4.1 may be used to bound $\text{ch}_\Sigma^e(G \square H)$ for many more graphs G and H .

We will use the following propositions on the tree width of a graph G , denoted $\text{tw}(G)$.

Proposition 4.2. *If H is a subgraph of G , then $\text{tw}(H) \leq \text{tw}(G)$.*

Proposition 4.3. *Every graph G has a vertex $v \in V(G)$ such that $d_G(v) \leq \text{tw}(G)$.*

Using these properties, we may bound $\text{ch}_\Sigma^e(G)$ if G is a particular Cartesian product of two graphs. Note that, for the graph $G \square H$ and vertex $v \in V(G)$, the subgraph induced by the set of vertices $\{(v, x) : x \in V(H)\}$ is denoted (v, H) .

Theorem 4.4. *Let H be a regular graph on $n \geq 3$ vertices which contains a 2-factor. If G is a graph with tree-width $\text{tw}(G) = t$, then (1) $\text{mind}(G \square H) \leq nt + \text{mind}(H)$ and (2) $\text{ch}_\Sigma^e(G \square H) \leq nt + \text{mind}(H) + 1$.*

Proof. We may assume that G is connected. The proof of (1) is by induction on $|V(G)|$; the statement is true when G is a single vertex, since $\text{tw}(G) = 0$ and $\text{ch}_\Sigma^e(H) \leq \text{mind}(H) + 1$ is guaranteed by Lemma 2.1.

Suppose $|V(G)| \geq 2$. By Corollary 4.3, there is a $v \in V(G)$ such that $d_G(v) \leq \text{tw}(G)$. Let X be the minimal edge cut for (v, H) . Since $|X| = n \cdot d_G(v)$ and $\text{tw}(G - v) \leq \text{tw}(G)$ by Proposition 4.2, Lemma 4.1 implies that

$$\begin{aligned} \text{mind}(G \square H) &\leq \max\{\text{mind}(H) + nd_G(v), \text{mind}((G \square H) - X)\} \\ &\leq \max\{\text{mind}(H) + nt, \text{mind}((G \square H) - (v, H))\} \\ &\leq \max\{\text{mind}(H) + nt, \text{mind}((G - v) \square H)\} \\ &\leq \max\{\text{mind}(H) + nt, n\text{tw}(G - v) + \text{mind}(H)\} \\ &\leq \text{mind}(H) + nt. \end{aligned}$$

Part (2) follows directly from Lemma 2.1. □

Since $\text{mind}(K_n)$ and $\text{mind}(C_n)$ are at most 2 (Corollary 3.4 and Proposition 3.5), the following corollary is obtained:

Corollary 4.5. *For any integer $n \geq 3$ and any graph G*

1. $\text{ch}_\Sigma^e(G \square K_n) \leq n \cdot \text{tw}(G) + 3$, and
2. $\text{ch}_\Sigma^e(G \square C_n) \leq n \cdot \text{tw}(G) + 3$.

In particular, if $G = T$ is a tree, then $\text{ch}_\Sigma^e(T \square K_n) \leq n + 1$ and $\text{ch}_\Sigma^e(T \square C_n) \leq n + 1$. Since $\Delta(T \square K_n)$ and $\Delta(T \square C_n)$ may be arbitrarily high for any fixed n , this gives a noticeable improvement on the bound from Theorem 3.1. More generally, if G is K_4 -minor free (equivalently, G is **series-parallel**), then $\text{tw}(G) = 2$ and so $\text{ch}_\Sigma^e(G \square K_n)$ and $\text{ch}_\Sigma^e(G \square C_n)$ are each at most $2n + 1$.

5 Additive colourings from list-weightings

In [6], a vertex weighting version of the “standard” weighting-colouring problem is considered – what is the smallest k such that every graph G has a vertex weighting $w : V(G) \rightarrow [k]$ which properly colours $V(G)$ by the colouring function $c(v) = \sum_{u \in N(v)} w(u)$? A vertex weighting which colours $V(G)$ in this way is called an **additive colouring** (originally called a “lucky labelling” in [6]). The **additive colouring number** of G , denoted $\chi_\Sigma^v(G)$, is the smallest k for which G has an additive colouring by $[k]$. The following conjecture is offered on additive colourings:

Additive Colouring Conjecture (Czerwiński, Grytczuk, Żelazny, [6]). *For any graph G , $\chi_\Sigma^v(G) \leq \chi(G)$.*

No constant bound is possible for the additive colouring number, since $\chi_\Sigma^v(K_n) = n$ for any n . However, it is shown in [4] that $\chi_\Sigma^v(G) \leq 468$ if G is planar, $\chi_\Sigma^v(G) \leq 36$ if G is planar and 3-colourable, and $\chi_\Sigma^v(G) \leq 4$ if G is planar and has girth at least 13. Furthermore, it is shown in [5] that, for any graph G , there exists a set S_G of $\chi(G)$ integers such that there exists a weighting $w : V(G) \rightarrow S_G$ which is an additive colouring.

Consider the usual list variation of this problem. Denote by $\text{ch}_\Sigma^v(G)$ the smallest k such that G has an additive colouring from any assignment of lists of size k to the vertices of G , and call $\text{ch}_\Sigma^v(G)$ the **additive choosability number** of G . The following conjecture is offered in the spirit of the List 1-2-3 and List 1-2 Conjectures:

Additive List Colouring Conjecture. *For any graph G , $\text{ch}_\Sigma^v(G) = \chi_\Sigma^v(G)$.*

One may directly try to show that $\text{ch}_\Sigma^v(G) \leq \chi(G)$, which would also prove the Additive Colouring Conjecture (since clearly $\text{ch}_\Sigma^v(G) \geq \chi_\Sigma^v(G)$).

First consider a simple class of graphs – complete graphs.

Proposition 5.1. *If $G = K_n$ for any $n \geq 2$, then $\text{ch}_\Sigma^v(G) = n$.*

Proof. It is well known that $\text{ch}(G) = n$; we will show additive list colourings of K_n are precisely proper list-vertex colourings of K_n .

Assign to each vertex a list of real numbers. Let ϕ be a vertex colouring of G from these lists, and let $c(v) = \sum_{u \in N_G(v)} \phi(u)$. Clearly $c(v) = \sum_{u \in N_G(v)} \phi(u) - \phi(v)$, and hence $c(u) \neq c(v) \iff \phi(u) \neq \phi(v)$ for $u, v \in V(G)$. \square

The best published universal bound on $\chi_\Sigma^v(G)$ is, in fact, an upper bound on the additive choosability number of a graph G :

Theorem 5.2 (Akbari, Ghanbari, Manaviyat, Zare [1]). *For any graph G , $\text{ch}_\Sigma^v(G) \leq \Delta(G)^2 - \Delta(G) + 1$.*

We can prove something that is, in a sense, slightly stronger:

Theorem 5.3. *If G is a d -degenerate graph, then $\text{ch}_\Sigma^v(G) \leq d\Delta(G) + 1$.*

Proof. Without loss of generality, assume that G is connected. The proof is by induction on $|V(G)|$. The statement is true if $|V(G)| \leq 3$. Let G be a graph on at least 4 vertices and let $v \in V(G)$ be a vertex with degree $d(v) \leq d$. By induction, there exists a vertex weighting of $G - v$ by lists which is an additive colouring of $G - v$. Note that the sum which colours v is now determined, and choosing $w(v)$ only changes the colour of the vertices $u \in N(v)$. Since each $u \in N(v)$ has at most $\Delta(G)$ neighbours in G , there are at most $d\Delta(G)$ forbidden values for $w(v)$ and so an admissible choice for $w(v)$ exists. \square

It the easily follows that $\text{ch}_\Sigma^v(G) \leq \Delta(G)(\chi(G) - 1) + 1$ if G is chordal, $\text{ch}_\Sigma^v(G) \leq 2\Delta + 1$ if G is K_4 -minor free, and $\text{ch}_\Sigma^v(G) \leq 5\Delta(G) + 1$ if G is planar. Finally, the theorem below shows that the Additive Colouring Conjecture is true for complete multipartite graphs, and in fact a stronger bound than conjectured is attainable for list-weightings.

Theorem 5.4. *If G is a complete multipartite graph K_{t_1, t_2, \dots, t_k} and $t = \min\{t_i : 1 \leq i \leq k\}$, then $\text{ch}_\Sigma^v(G) \leq \lceil \frac{k-1}{t} \rceil + 1$.*

Proof. Let V_i denote the colour class of size t_i . Let \mathcal{L} be a list-assignment of $V(G)$ such that, for each $1 \leq i \leq k$, each $v \in V_i$ receives a list L_v of size $|L_v| = \lceil \frac{k-1}{t_i} + 1 \rceil$; note that $|L_v| \leq \lceil \frac{k-1}{t} + 1 \rceil$ for every $v \in V(G)$.

Let $e = v_i v_j \in E(G)$, where $v_i \in V_i, v_j \in V_j$. For a list-weighting w into \mathcal{L} , denote $w(V_i) = \sum_{x \in V_i} w(x)$. For any weighting function w from \mathcal{L} and resulting vertex colouring by sums c ,

$$\begin{aligned} c(v_i) \neq c(v_j) &\iff \sum_{x \in N_G(v_i)} w(x) \neq \sum_{y \in N_G(v_j)} w(y) \\ &\iff \sum_{x \in V(G)} w(x) - w(V_i) \neq \sum_{y \in V(G)} w(y) - w(V_j) \\ &\iff w(V_i) \neq w(V_j) \end{aligned}$$

It suffices then to find a weighting function w such that $w(V_i) \neq w(V_j)$ for each $1 \leq i, j \leq k$.

The minimum number of possible sums attainable by choosing one element from m lists of size r is $mr - m + 1$, and hence there are at least $t_i \left\lceil \left(\frac{k-1}{t_i} + 1 \right) - 1 \right\rceil + 1 \geq k$ attainable values of $w(V_i)$. Let L_i denote this list of k attainable values. Let H be a complete graph having vertex set $V(H) = \{V_1, \dots, V_t\}$, and assign to each V_i the list L_i . Since $\text{ch}_\Sigma^v(K_k) = k$ by Proposition 5.1, there exists an additive colouring of H given by some list-weighting $w' : V(H) \rightarrow \cup_{1 \leq i \leq k} L_k$. For each set $V_i \subset V(G)$, there exists a function $w_i : V_i \rightarrow \cup_{x \in V_i} L_x$ such that $w'(V_i) = \sum_{x \in V_i} w_i(x)$. Letting $w(v_i) = w_i(v_i)$ if $v_i \in V_i$ thus gives $w(V_i) = w'(V_i)$ for each $1 \leq i \leq k$, and hence the desired additive colouring of G . \square

Note that in the case of regular complete k -partite graphs, this implies that $\text{ch}_\Sigma^v(G) \leq \left\lceil \frac{\chi(G)-1}{\alpha(G)} + 1 \right\rceil$. This generalizes a result of Chartrand, Okamoto, and Zhang [5], who show that $\chi_\Sigma(G) = \left\lceil \frac{\chi(G)-1}{\alpha(G)} + 1 \right\rceil$ for such graphs only in the case when each part has size $\binom{n+l-1}{l-1}$ for positive integers n and $l \geq 2$.

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